GENOME-SCALE ALGORITHM DESIGN
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Exercises for Chapter 5. Network flows

5.1 Given an arbitrary graph G with a unique source s and a unique sink t, and given a function $f : E(G) \to \mathbb{Q}$ satisfying the flow conservation property on G, show that

$$\sum_{y \in N^+(s)} f(s, y) = \sum_{y \in N^-(t)} f(y, t).$$

5.2 Let G be a DAG with a unique source s and a unique sink t, and let $f : E(G) \to \mathbb{N}$ satisfy the flow conservation property on G, such that the flow exiting s equals q. Show that if there exists an integer x such that, $f(e)$ is a multiple of x, for every $e \in E(G)$, then f can be decomposed into q/x paths, each of weight x.

5.3 Show that a flow f on an arbitrary graph G with a unique source s and a unique sink t can be decomposed into at most $|E(G)|$ weighted paths or cycles. How fast can this be done?

5.4 In the 3-partition problem, we are given a set $A = \{a_1, \ldots, a_{3q}\}$ of $3q$ positive integers, such that

- $\sum_{i=1}^{3q} a_i = qB$, where B is an integer, and
- for all $i \in \{1, \ldots, 3q\}$ it holds that $B/4 < a_i < B/2$.

We are asked whether there exists a partition of A into q disjoint sets, such that the sum of the integers in each of these sets is B. This problem is known to be NP-hard, even when the values of the $3q$ integers in the set A are bounded by a certain value not depending on q (it is called *strongly* NP-hard). Use a similar reduction to that in the proof of Theorem 5.3 to show that the 3-Partition problem can be reduced to the problem of decomposing a flow in a DAG into the minimum number of weighted paths.

5.5 Given a weighted DAG G, show that we can find an s-t path whose bottleneck is maximum among all s-t paths of G, in time $O(|E(G)|)$.

5.6 Suppose that in a flow network N with a unique source and a unique sink some arcs have infinite capacity. Show that there exists a flow f over N of minimum cost, with the additional property that the value of f on each arc without capacity is at most the sum of all arc capacities.

5.7 Show that the minimum-cost circulation problem can be reduced to the minimum-cost flow problem.

5.8 Given a circulation f over a circulation network N, show that if C is a cycle in the residual graph $\mathsf{R}(f)$ of f, then the circulation f_C defined in Section 5.2 is a circulation over N.

5.9 Given a circulation network $N = (G, \ell, u, c)$, consider the problem of finding a feasible circulation over N, that is, a circulation satisfying the demand and capacity constraints of N. Construct the circulation network N' as follows:

- N' has the same vertices as N;
- for every arc (x, y) of N, add to N' two parallel arcs with demand 0: one with cost -1 and capacity $\ell(x, y)$, and one with cost 0 and capacity $u(x, y) - \ell(x, y)$.

Show that N admits a feasible circulation if and only if the minimum-cost circulation over N' has cost less than or equal to minus the sum of all demands of N. How can you obtain a feasible circulation f over N from a minimum-cost circulation f' over N'?

5.10 Show that the reduction of a minimum-cost flow problem with convex costs to a standard minimum-cost flow problem from Insight 5.2 is correct.

5.11 Given a DAG G with a unique source s and a unique sink t, show that we can find the maximum number of s-t paths without common vertices (apart from s and t) by a reduction to a maximum flow problem.

5.12 In the *minimum flow problem*, we are given a flow network N with arc demands only, and are asked to find a flow of minimum value over N satisfying all demand constraints. Show that the minimum flow problem can be solved by two applications of a maximum flow problem.

5.13 Show that finding a *maximum-cardinality matching* in a bipartite graph can be reduced to a minimum-cost flow problem.

5.14 Show that, among all maximum-cardinality matchings of a bipartite graph G, the problem of finding one of minimum-cost can be reduced to a minimum-cost network flow problem.

5.15 Suppose that you have an algorithm for solving the minimum-cost perfect matching problem in an arbitrary graph (not necessarily bipartite). Show that, given any graph G with costs associated with its edges, you can use this algorithm, by appropriately transforming G, to find a *minimum-cost maximum matching* (that is, a matching of maximum cardinality in G, with the additional property that among all matchings of maximum cardinality, it has minimum cost).

5.16 Consider Problem 5.7 in which one is required to minimize instead

$$\sum_{(x,y)\in M} c(x, y) + \alpha \sum_{y\in B} |\mathtt{id}(y) - d_M(y)|^2 .$$

How do you need to modify the reduction given for Problem 5.7 to solve this new problem? Does the resulting flow network still have size polynomial in the size of the input? *Hint.* Use the idea from Insight 5.2.

5.17 Show that the problem of finding a minimum-cost disjoint cycle cover in an undirected graph can be reduced to the problem of finding a minimum-cost disjoint cycle cover in a directed graph.

5.18 An *edge cover* of an undirected graph G is a set of edges covering all the vertices of G.

- Show that a minimum-cost edge cover in a bipartite graph can be solved by a minimum-cost flow problem.
- Show that a minimum-cost edge cover in an arbitrary graph G can be reduced to the problem of finding a maximum-weight matching in G.

Additional exercises not in the book

5.19 Give an example of a directed acyclic graph which admits two decomposition into weighted paths, each with a different number of paths.

5.20 Give an example of a directed graph G (not acyclic) with unique source s and unique sink t, and of a flow f in G, such that f cannot be decomposed into a set of weighted s-t paths.

5.21 Given a directed graph G with a unique source s and unique sink t, in which every arc (x, y) has a capacity $u(x, y)$, construct a minimum-cost flow network N in which every arc (x, y) of G has demand 0, capacity $u(x, y)$ and cost -1. Add the arc (s, t) with demand 0, capacity ∞ and cost 0, and set the flow value to ∞. Give an example which disproves the following claim: the minimum-cost flow in N is a maximum flow in G with capacity function u.

5.22 Give an example of a bipartite graph G, and a non-integer-valued flow f, feasible for the network built for the minimum-cost perfect matching problem on G, such that f does not induce a matching in G.

5.23 Give an example of a directed acyclic graph G and of a non-integer-valued flow f, feasible for the network built for the minimum-cost minimum path cover problem on G, such that f does not induce an optimal solution in G.

5.24 Consider the following simpler variant of the minimum-cost minimum path cover problem. Given a DAG $G = (V, E)$ with unique source s and unique sink t, a cost function $c : E \to \mathbb{Q}_+$, and an integer k, find k s-t paths $P_1, \ldots, P_k$ forming a path cover of G (i.e. each vertex of G belongs to some P_i) and minimizing $\sum_{i=1}^{k} \sum_{(x,y) \in P_i} c(x, y)$. Solve this problem by a reduction to a minimum-cost flow problem.

5.25 Consider the following problem: given a directed graph G, find the minimum number of paths (possibly with repeated vertices) that cover all the vertices of G. Give an example of a directed graph where this number is two. Show that this problem can be reduced to a minimum-cost flow problem.